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Application of the generalized differential quadrature rule to initial–boundary-value problems

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Abstract

Partial differential equations (PDEs) for the forced vibration of structural beams are solved in this paper using the recently proposed generalized differential quadrature rule (GDQR). The GDQR techniques are first applied to both spatial and time dimensions simultaneously as a whole. No other classical methods are needed in the time dimension. The objective of this paper is to formularize the GDQR expressions and corresponding explicit weighting coefficients, while the derivation of explicit weighting coefficients is one of the most important aspects in the differential quadrature methods. It should be emphasized that the GDQR expressions and weighting coefficients for two-dimensional problems are not a direct application of those for one-dimensional problems, and they are distinctly different for PDEs of different orders. An Euler beam and a Timoshenko beam are employed as examples. Accurate results are obtained. The proposed procedures can be applied to problems in other disciplines of sciences and technology, where the problems may be governed by other PDEs with different orders in the time or spatial dimension.

1. Introduction

The differential quadrature method (DQM) was proposed in early 1970s in order to solve partial, i.e., initial–boundary value, differential equations [1,2]. The DQM is usually applied only in the spatial dimension. Classical methods, such as Runge–Kutta methods, are used in the time dimension. The DQM has never been implemented in the time dimension of PDEs when the temporal order is second order or higher. It seems that Tomasiello [3] has coped with

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initial–boundary-value problems for the forced vibration of Euler beams. However, he did not implement the DQM in the time dimension. The present authors have proposed the generalized differential quadrature rule (GDQR) to deal with initial−value ordinary differential equations (ODEs) of second through fourth orders [4,5] and boundary-value problems of fourth−, sixth−, and eighth orders [6–9] in solid mechanics and of third and sixth orders in fluid mechanics [10]. The authors have also applied the conventional DQM to the problems with their boundary conditions specified at three and four distinct points [11]. Partial differential equations for classical rectangular plates are of fourth order in both spatial dimensions [7]. The PDEs for the forced vibration of some structural beams are of second order in the time dimension and of fourth order in the spatial dimension. It should be emphasized that the GDQR expressions for differential equations of different orders are distinctly different. The objective of this paper is to extend the GDQR to a new application of initial–boundary-value problems and to derive corresponding explicit weighting coefficients, while the derivation of explicit weighting coefficients is one of the most important aspects for an accurate implementation of the DQMs. The explicit weighting coefficients in the conventional DQM have been obtained using the Lagrange interpolation functions in Refs. [12–16].

The forced vibration of some structural beams is solved here in a completely original way by applying the GDQR to both time and spatial dimensions simultaneously as a whole. The GDQR expressions and corresponding explicit weighting coefficients have been derived for the first time. The GDQR can obtain very accurate results using only a few sampling points. An Euler beam and a Timoshenko beam are used as examples. Accurate results are obtained. In other disciplines of sciences and technology, there are enormous other high−order initial–boundary−value problems with different orders in the time or spatial dimension, which could be dealt with according to the procedures proposed here and in paper [7]. It is quite evident that the GDQR expressions and weighting coefficients for two-dimensional problems are not a direct application of those for one-dimensional problems.

2. Reference problems

The governing equation for the forced vibration of an Euler beam is expressed as [17]

$$EI\frac{\partial^4 y}{\partial x^4} + \rho\frac{\partial^2 y}{\partial t^2} = Q\sin\frac{\pi x}{L}\sin pt, \tag{1}$$

where $EI = 4.7726 \times 10^7$ is the stiffness coefficient, $\rho = 420$ the mass density per unit length, $Q = 10^7$ the maximum force, $L = 10$ the beam length, and $p = 2\pi/0.28335$ the frequency of the dynamic force.

Both the spatial domain [0, L] and time domain [0, T] are transformed to [0, 1], using $X = x/L$ and $\tau = t/T$. Eq. (1) is then non-dimensionlized as

$$\frac{EI}{L^4}\frac{\partial^4 y}{\partial X^4} + \frac{\rho}{T^2}\frac{\partial^2 y}{\partial \tau^2} = Q\sin\pi X\sin pT\tau. \tag{2}$$

If the beam is simply supported at both ends, the boundary conditions are:

$$y = 0, \quad \partial^2 y/\partial^2 X = 0 \quad (X = 0, 1,\ \tau \geq 0). \tag{3}$$

The initial conditions are as follows:

$$y = 0, \quad \partial y/\partial \tau = 0 \quad (\tau = 0). \tag{4}$$

The analytic solutions for the displacement and bending moment are obtained as [17]

$$y = \frac{Q}{\rho}\sin\left(\frac{\pi x}{L}\right)\frac{\sin pt - (p/\omega)\sin\omega t}{\omega^2 - p^2} \tag{5}$$

$$M = EI\frac{\partial^2 y}{\partial x^2} = -\frac{QEI\pi^2}{\rho L^2}\sin\left(\frac{\pi x}{L}\right)\frac{\sin pt - (p/\omega)\sin\omega t}{\omega^2 - p^2}, \tag{6}$$

where

$$\omega = \pi^2\sqrt{\frac{EI}{\rho L^4}}. \tag{7}$$

Eq. (2) does not contain terms of mixed derivatives, while PDEs for classical rectangular plates do [7]. The difference between the conventional DQM and the proposed GDQR is manifested in the differential quadrature expressions for mixed derivatives. To illustrate the application of a complete version of the GDQR to initial-boundary–value problems, a Timoshenko beam with a mixed derivative term is used as the other example. The PDE for a Timoshenko beam considering the effect of rotary inertia or shear deformation can be expressed as [17]

$$\frac{EI}{L^4}\frac{\partial^4 y}{\partial X^4} + \frac{\rho}{T^2}\frac{\partial^2 y}{\partial \tau^2} - \frac{\zeta\rho}{T^2L^2}\frac{\partial^4 y}{\partial X^2 \partial \tau^2} = Q \sin \pi X \sin pT\tau, \tag{8}$$

where ζ is a constant related to the effect of rotary inertia or shear deformation. The corresponding frequency is [17]

$$\omega = \pi^2\sqrt{\frac{EI}{\rho L^4(1+\zeta\pi^2)}}. \tag{9}$$

If the boundary and initial conditions also adopt Eqs. (3) and (4), its corresponding analytical solutions are written as

$$y = \frac{Q}{\rho(1+\zeta\pi^2)}\sin\left(\frac{\pi x}{L}\right)\frac{\sin pt - (p/\omega)\sin\omega t}{\omega^2 - p^2}. \tag{10}$$

$$M = EI\frac{\partial^2 y}{\partial x^2} = -\frac{QEI\pi^2}{\rho L^2(1+\zeta\pi^2)}\sin\left(\frac{\pi x}{L}\right)\frac{\sin pt - (p/\omega)\sin\omega t}{\omega^2 - p^2}. \tag{11}$$

The data in the later numerical analysis are as follows:

$$T = 0.25, \quad \zeta = 0.3/\pi^2 \tag{12}$$

3. Formulation

The GDQR expressions and weighting coefficients for PDEs will be constructed with the help of those for ODEs. The conventional DQM's expression for ODEs is written as, if the Lagrange interpolation shape functions $l_j(x)$ are used as trial functions

$$\left.\frac{\mathrm{d}^r y}{\mathrm{d}x^r}\right|_{x=x_i} = \sum_{j=1}^{N} A_{ij}^{(r)} y_j = \sum_{j=1}^{N} l_j^{(r)}(x_i) y_j \quad (i = 1, 2, \ldots, N), \tag{13}$$

where N is the number of all the discrete sampling points. $A_{ij}^{(r)}$ are the weighting coefficients of the rth order derivative of the function $y(x)$ associated with points x_i. $l_j^{(r)}(x_i)$ is the rth order derivative, and $A_{ij}^{(r)} = l_j^{(r)}(x_i)$ were obtained in Refs. [12–16].

The GDQR expression for a two–point boundary–value fourth order ODE is expressed as [6,7]

$$\left.\frac{\mathrm{d}^r y}{\mathrm{d}x^r}\right|_{x=x_i} = \sum_{j=1}^{N+2} E_{ij}^{(r)} W_j = \sum_{j=1}^{N+2} h_j^{(r)}(x_i) W_j \quad (i = 1, 2, \ldots, N), \tag{14}$$

where $\{W\}^{\mathrm{T}} = \left\{ y_1^{(1)}, y_1, y_2, y_3, \ldots, y_{N-1}, y_N, y_N^{(1)} \right\}$. $h_j(x)$ are the corresponding Hermite–Fejér interpolation functions. $E_{ij}^{(r)}$ are the GDQR's weighting coefficients of the rth order derivative of the function $y(x)$ at point x_i, and $E_{ij}^{(r)} = h_j^{(r)}(x_i)$ have been applied to Euler beam analysis in Refs. [6,7].

The GDQR expression for an initial–value second order ODE is expressed as [4,5]

$$\left.\frac{\mathrm{d}^r y}{\mathrm{d}\tau^r}\right|_{\tau=\tau_i} = \sum_{j=1}^{N+1} F_{ij}^{(r)} Q_j = \sum_{j=1}^{N+1} p_j^{(r)}(\tau_i) Q_j \quad (\tau = 1, 2, \ldots N), \tag{15}$$

where $\{Q\}^{\mathrm{T}} = \left\{ y_1, y_2, \ldots, y_N, y_N^{(1)} \right\}$. $p_j(x)$ are the corresponding Hermite–Fejér interpolation functions. $F_{ij}^{(r)}$ are the GDQR's weighting coefficients of the rth order derivative of the function $y(x)$ at point τ_i, and $F_{ij}^{(r)} = p_j^{(r)}(\tau_i)$ have been applied to one–degree–of–freedom dynamic problems in Refs. [4,5], for example, Duffing equations. Note that the notations for both $p_j(\tau)$ and $F_{ij}^{(r)}$ are different from those in Refs. [4,5] in order to avoid confusion between Eqs. (14) and (15). It should be noted that the inverse node numbering is used in the initial value problems for a programming convenience. It means that y_N and $y_N^{(1)}$ are initial conditions [4,5].

The Hermite–Fejér interpolation functions $h_j(x)$ and $p_j(\tau)$, and notations for weighting coefficients $E_{ij}^{(r)}$ and $F_{ij}^{(r)}$ for ODEs will be employed in these serial numbers to derive the corresponding weighting coefficients for initial–boundary–value PDEs. The construction procedures are similar to those for rectangular plate problems [7] and are described below in detail for the clarity of this paper.

The key point for choosing independent variables for PDEs is that the number of independent variables at a point is equal to the number of equations/conditions to be satisfied at the same point. Eqs. (2) and (8) are similar in the eye of the GDQR, since both of them are of second order in the time dimension and of fourth order in the spatial dimension. Both the spatial and time dimensions are discretized simultaneously in Fig. 1. Next, the independent variables for each point are chosen according to the GDQR's definitions, as shown in Table 1(a). At the spatial dimension ends, both the function value and its first order derivative are used as independent variables at the two ends, as was done for one-dimensional beam problems in Refs. [6,7]. At two temporal dimension ends, initial displacement and velocity are used as independent variables at the initial point, and only displacement is employed as a independent variable at the time end point where only governing equation needs satisfying. The symbol U_{ij} in Table 1(a) is used as the replacement

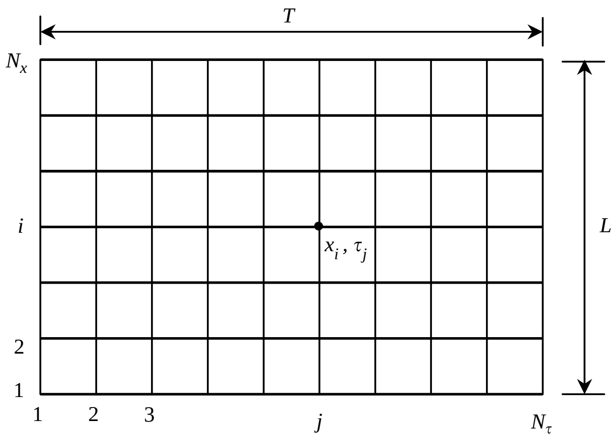

Fig. 1. The GDQR's grids for both the spatial and time domains.

Table 1
(a) The independent variables, and (b) interpolation shape functions, which have a one-to-one correspondence with each other for the forced vibration of beams

(a) *The independent variables*				
$\partial y_{N_x 1}/\partial X = U_{(N_x+2)1}$	$\partial y_{N_x 2}/\partial X = U_{(N_x+2)2}$	…	$\partial y_{N_x N_\tau}/\partial X = U_{(N_x+2)N_\tau}$	Nil
$y_{N_x 1} = U_{(N_x+1)1}$	$y_{N_x 2} = U_{(N_x+1)2}$	…	$y_{N_x N_\tau} = U_{(N_x+1)N_\tau}$	$\partial y_{N_x N_\tau}/\partial \tau = U_{(N_x+1)(N_\tau+1)}$
…	…	…	…	…
$y_{21} = U_{31}$	$y_{22} = U_{32}$	…	$y_{2N_\tau} = U_{3N_\tau}$	$\partial y_{2N_\tau}/\partial \tau = U_{3(N_\tau+1)}$
$y_{11} = U_{21}$	$y_{12} = U_{22}$	…	$y_{1N_\tau} = U_{2N_\tau}$	$\partial y_{1N_\tau}/\partial \tau = U_{2(N_\tau+1)}$
$\partial y_{11}/\partial X = U_{11}$	$\partial y_{12}/\partial X = U_{12}$	…	$\partial y_{1N_\tau}/\partial X = U_{1N_\tau}$	Nil
(b) *The interpolation shape functions*				
$l_1(\tau)h_{N_x+2}(X)$	$l_2(\tau)h_{N_x+2}(X)$	…	$l_{N_\tau}(\tau)h_{N_x+2}(X)$	Nil
$p_1(\tau)h_{N_x+1}(X)$	$p_2(\tau)h_{N_x+1}(X)$	…	$p_{N_\tau}(\tau)h_{N_x+1}(X)$	$p_{(N_\tau+1)}(\tau)l_{N_x}(X)$
…	…	…	…	…
$p_1(\tau)h_3(X)$	$p_2(\tau)h_3(X)$	…	$p_{N_\tau}(\tau)h_3(X)$	$p_{(N_\tau+1)}(\tau)l_2(X)$
$p_1(\tau)h_2(X)$	$p_2(\tau)h_2(X)$	…	$p_{N_\tau}(\tau)h_2(X)$	$p_{(N_\tau+1)}(\tau)l_1(X)$
$l_1(\tau)h_1(X)$	$l_2(\tau)h_1(X)$	…	$l_{N_\tau}(\tau)h_1(X)$	Nil

of independent variables just for convenient formulations. The Hermite-Fejér interpolation functions for each independent variable are shown in Table 1(b). The properties of the Hermite–Fejér interpolation functions can be easily verified. Therefore, the interpolation expression of the displacement function are summarized as

$$y(X,\tau) = \sum_{k=1}^{N\tau}\sum_{m=2}^{N_x+1} p_k(\tau)h_m(X)\cdot U_{mk} + \sum_{k=1}^{N_\tau} l_k(\tau)\big(h_1(X)U_{1k} + h_{N_x+2}(X)\cdot U_{(N_x+2)k}\big)$$
$$+ \sum_{k=1}^{N_x} l_k(X)p_{(N_\tau+1)}(\tau)\cdot U_{(k+1)(N_\tau+1)}. \tag{16}$$

Using Eq. (16), the GDQR's expression for the rth order $X-$partial derivative at points $X = X_i$ along any line $\tau = \tau_j$ parallel to the $X-$axis may be written as

$$\left.\frac{\partial^r y}{\partial X^r}\right|_{X=X_i,\tau=\tau_j} = \sum_{k=1}^{N_x+2} E_{ik}^{(r)} U_{kj} \tag{17}$$

for the sth order $\tau-$partial derivative at points $\tau = \tau_j$ along any line $X = X_i$ parallel to the $\tau-$axis may be written as

$$\left.\frac{\partial^s y}{\partial \tau^s}\right|_{X=X_i,\tau=\tau_j} = \sum_{k=1}^{N_\tau+1} F_{jk}^{(s)} U_{ik}. \tag{18}$$

The GDQR's expression for a mixed derivative at point (X_i, τ_j) is in the form of

$$\begin{aligned}\left.\frac{\partial^{(r+s)} y}{\partial^r X \partial^s \tau}\right|_{X=X_i,\tau=\tau_j} = & \sum_{k=1}^{N\tau} \sum_{m=2}^{N_x+1} F_{jk}^{(s)} E_{im}^{(r)} U_{mk} \\ & + \sum_{k=1}^{N_\tau} A_{jk}^{(s)} \left(E_{i1}^{(r)} U_{1k} + E_{i(N_x+2)}^{(r)} U_{(N_x+2)k} \right) \\ & + \sum_{k=1}^{N_x} A_{ik}^{(r)} F_{j(N_\tau+1)}^{(s)} U_{(k+1)(N_\tau+1)}.\end{aligned} \tag{19}$$

Eqs. (17)–(19) show clearly that the GDQR expressions are derived from the interpolation function of Eq. (16), as done for the classical rectangular plate problems [7]. However, The difference between PDEs of different orders is manifested in the differential quadrature expressions for mixed derivatives. This means that both the GDQR expressions and weighting coefficients are quite different.

Using the equations derived, governing equation (8) is discretized as

$$\begin{gathered}\frac{EI}{L^4} \sum_{k=1}^{N_x+2} E_{ik}^{(4)} U_{kj} + \frac{\rho}{T^2} \sum_{k=1}^{N_\tau+1} F_{jk}^{(2)} U_{ik} - \frac{\zeta\rho}{T^2 L^2} \\ \left[\sum_{k=1}^{N_\tau} \sum_{m=2}^{N_x+1} F_{jk}^{(2)} E_{im}^{(2)} U_{mk} + \sum_{k=1}^{N\tau} A_{jk}^{2} \left(E_{i1}^{(2)} U_{1k} + E_{i(N_x+2)}^{(2)} U_{(N_x+2)k} \right) \right. \\ \left. + \sum_{k=1}^{N_x} A_{ik}^{(2)} F_{j(N_\tau+1)}^{(2)} U_{(k+1)(N_\tau+1)} \right] = Q \sin \pi X_i \sin pT\tau_j \\ (i = 3, \ldots, N_x; j = 1, 2, \ldots, N_\tau - 1).\end{gathered} \tag{20}$$

Boundary Eq. (3) can be transformed to

$$U_{ij} = 0, \sum_{k=1}^{N_x+2} E_{ik}^{(2)} U_{kj} = 0 \quad (j = 1, 2, \ldots, N_\tau). \tag{21}$$

Initial condition equation (4) may be written as

$$U_{iN_\tau} = 0, U_{j(N_\tau+1)} = 0 \quad (\mathrm{i} = 1, 2, \ldots, N_x + 2; j = 2, 3, \ldots, N_x + 1). \tag{22}$$

The formed algebraic equations from (20)–(22) can be solved to obtain the required independent variables. Using Eq. (17), the bending moment at any point can be obtained. The velocity and acceleration are calculated using Eq. (18). If ζ is taken as zero in Eq. (20), the results for the Euler problem can be obtained.

4. Results and discussion

Chebyshev–Gauss–Lobatto sampling points are employed in this work. Using only seven points in the spatial domain and thirteen points in the time domain, the GDQR's relative errors for the Euler beam are shown in Tables 2 and 3, and those for the Timoshenko beam in Tables 4 and 5. Only the results in half of the spatial domain are shown due to its symmetric nature. It is seen that the GDQR results are very accurate. It is clearly shown from this work that the DQ techniques can directly transform PDEs to discrete algebraic equations. No classical methods, such as Runge–Kutta methods, are needed in any dimension. This is an apparent advantage for the DQ techniques. The procedures here can be applied to the forced vibration of circular plates if the time dimension is added in Refs. [18,19], since their governing equations can all be expressed as the PDEs with the fourth order in spatial dimension and the second order in temporal dimension. As compared with the FDM, the accuracy of the GDQR is greatly manifested in Ref. [20].

The GDQR is demonstrated here to solve high order PDEs without using the three conventional techniques such as building the boundary conditions into weighting coefficients, dropping equations at points closest to the domain ends, and the δ-point techniques. The classical rectangular plate problem [7] is a fourth order boundary-value PDE with two boundary conditions at each end of its four boundaries. The forced vibration of beams is a PDE of the fourth order in the spatial dimension and the second order in the time dimension. Independent variables for these two problems are different. The explicit weighting coefficients are derived for an easy and accurate implementation.

In the GDQR, one needs some discernment in the discovery of the number of the equations at a point in multiple-dimensional problems. The classical rectangular plate has one, two and three equations at inner, boundary line and corner points, respectively [7]. The forced vibration of beams has one and two equations at inner and boundary line points, respectively. For the four corner points as shown in Fig. 1, the two points at the initial line have three equations and three

Table 2
Analytical displacements and corresponding relative errors for the Euler beam

	t_1	t_3	t_5	t_7	t_9	t_{11}
Analytical values of displacements						
X_2	−10.269	−12.610	−6.6302	7.5054	3.2406	0.084448
X_3	−34.759	−42.684	−22.443	25.406	10.969	0.28585
X_4	−49.157	−60.364	−31.740	35.929	15.513	0.40426
Corresponding relative errors (%)						
X_2	0.00337	0.00253	0.00022	0.00234	−0.00079	−0.03876
X_3	0.00288	0.00203	−0.00028	0.00184	−0.00128	−0.03914
X_4	0.00241	0.00157	−0.00074	0.00138	−0.00173	−0.03948

Table 3
Analytical bending moments and corresponding relative errors for the Euler beam

	t_1	t_3	t_5	t_7	t_9	t_{11}
Analytical values of bending moments ($\times 10^8$)						
X_2	0.48367	0.59394	0.31230	−0.35352	−0.15264	−0.0039777
X_3	1.6372	2.0105	1.0571	−1.1967	−0.51668	−0.013465
X_4	2.3155	2.8434	1.4951	−1.6924	−0.73072	−0.019042
Corresponding relative errors (%)						
X_2	0.00725	0.00640	0.00407	0.00622	0.00301	−0.03629
X_3	0.00442	0.00358	0.00126	0.00339	0.00024	−0.03807
X_4	0.00108	0.00024	−0.00206	0.00005	−0.00303	−0.04033

Table 4
Analytical displacements and corresponding relative errors for the Timoshenko beam

	t_1	t_3	t_5	t_7	t_9	t_{11}
Analytical values of displacements						
X_2	−14.019	−13.567	−3.1762	7.7598	2.6291	0.065174
X_3	−47.454	−45.925	−10.751	26.267	8.8994	0.22061
X_4	−67.110	−64.948	−15.205	37.147	12.586	0.31199
Corresponding relative errors (%)						
X_2	0.00182	0.00133	−0.00194	0.00147	−0.00043	−0.03729
X_3	0.00134	0.00081	−0.00243	0.00097	−0.00005	−0.00744
X_4	0.00090	0.00032	−0.00290	0.00050	−0.00050	−0.00753

Table 5
Analytical bending moments and corresponding relative errors for the Timoshenko beam

	t_1	t_3	t_5	t_7	t_9	t_{11}
Analytical values of bending moments ($\times 10^8$)						
X_2	0.66032	0.63905	0.14960	−0.36550	−0.12383	−0.0030699
X_3	2.2352	2.1632	0.50641	−1.2372	−0.41919	−0.010392
X_4	3.1612	3.0594	0.71620	−1.7498	−0.59284	−0.014696
Corresponding relative errors (%)						
X_2	0.00545	0.00550	0.00194	0.00542	0.00409	−0.00748
X_3	0.00280	0.00246	−0.00088	0.00254	0.00142	−0.00737
X_4	0.00134	−0.00111	−0.00423	−0.00086	−0.00175	−0.00740

independent variables, while the two points at the time-domain end have only two equations and two independent variables. Failing to find the number of the equations and thus the proper independent variables will not achieve the anticipated results. The principle about the choice of independent variables is formularized here, so is the procedure for the derivation of weighting coefficients. From the formulation of the GDQR proposed here and in paper [7] for classical rectangular plates, an application of the GDQR to other high order initial–boundary–value problems can be expected. The GDQR provides a way to directly implement the multiple given conditions and thus holds a great potential to much more practical problems.

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